Completeness of quantum theory and entanglement: A simplified teaching framework

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Abstract
This work proposes a clear teaching framework for the completeness of quantum theory and the related phenomenon of quantum entanglement. The framework begins by summarizing and underlining the probabilistic structure of quantum theory. Next, the well-known Einstein–Podolsky and Rosen (EPR) paradox, which questions the probabilistic structure and completeness of the theory, is summarized. Afterward, Bell's inequality and Clauser, Horne, Shimony, and Holt (CHSH) inequality approaches are enlightened by underlining their abilities to test completeness. The effort next explains how the completeness of quantum theory can be tested via quantum entanglement. Finally, the quantum entanglement of bipartite systems is generally illuminated, and the quantum entanglement of spins is fully resolved for undergraduate teaching purposes. The opinions of the students on the teaching proposal reveal that necessity, simplicity, and originality are high, and significance and importance are medium. The proposed teaching framework is reasonably simple and includes almost all the details that can be easily introduced for undergraduate quantum mechanics courses.

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1. Introduction

The Physics Education Research (PER) aims to enhance students' comprehension level and widen content by introducing new teaching materials and alternative teaching methods in accordance with scientific and technological developments. Quantum mechanics is surely at the forefront of PER because it is the base of numerous scientific and technological achievements. Quantum theory is foundationally based on operators used to define dynamic variables and wave/state functions to describe the behavior of quantum systems and therefore entails many counterintuitive and bizarre features such as probabilistic structure, quantization, the superposition principle, wave-particle duality, and uncertainty. The probabilistic structure of the theory is one of those problematic teaching issues that was, in fact, severely criticized by the founders of the theory. The probabilistic structure is questioned by the well-known Einstein-Podolsky-Rosen (EPR) paradox, which was later tackled by Bell's inequality and the CHSH inequality, both suggesting a route to test the probabilistic structure. These inequalities are later employed to test the probabilistic structure and therefore the completeness of the theory by employing quantum entanglement phenomena, which proves that the quantum theory is non-local and complete. Accordingly, it seems essential to include the completeness of theory and the quantum entanglement phenomenon in order to improve teaching quantum mechanics.

Teaching quantum mechanics is a challenge for educators because it contains many problematic concepts that should be fully resolved for teaching purposes (Müller & Wiesner, 2002). Numerous studies have been conducted to improve the teaching contents of quantum mechanics. Recently, an alternative approach to quantum physics teaching through analogy experiments has been reported (Aehle et al., 2022). In another study, teaching introductory quantum mechanics in terms of goals and practices was reported (Baily & Finkelstein, 2015). Carr and McKagan focused on quantum mechanics reform at the graduate level (Carr & McKagan, 2009). Colletti reported a study on an inclusive approach to teaching quantum mechanics in secondary schools (Colletti, 2003). The teaching superposition principle was considered a fundamental teaching concept in quantum theory (Ghirardi et al., 1995). There have been a number of studies focusing on improving undergraduate introductory quantum mechanics courses (Hobson, 1996; Hohenberg, 2010; Kohnle et al., 2013;

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Müller & Wiesner, 2002). There have also been some studies on introducing quantum mechanics at secondary school levels (Michelini et al., 2000; Olsen, 2002). In another study, current issues related to teaching quantum mechanics have likewise been undertaken (Rodríguez et al., 2017).

The probabilistic structure and completeness of quantum theory together with related topics, including quantum entanglement, are considered highly challenging. Quantum entanglement is, on the other hand, one of the most important features of theory and has attracted huge interest both scientifically and technologically (Horodecki et al. 2009). Hence, it is necessary to introduce the quantum entanglement phenomenon to undergraduate quantum mechanics textbooks and quantum mechanics courses worldwide. There have been obvious efforts to teach quantum entanglement and related topics, such as the EPR paradox and Bell' inequality. For instance, Pospiech addressed the EPR paradox in high schools (Pospiech, 1999). Dehlinger and Mitchell studied entangled photons, nonlocality, and Bell inequalities in undergraduate laboratories (Dehlinger & Mitchell, 2002). In another study, Shegelski focused on the topic of triplet versus singlet entangled states in teaching the Einstein–Podolsky–Rosen paradox (Shagelski, 2013). Griffiths also addressed the topics of EPR, Bell, and quantum locality (Griffiths, 2011). Additionally, the EPR paradox, Bell's inequality, and the question of locality were tackled by Blaylock (Blaylock, 2010). Recently, there have been some efforts to include quantum entanglement and related topics in quantum mechanics textbooks; nevertheless, the levels of the analysis are high and very difficult for undergraduate students (Wilde, 2011; Cohen-Tannoudji et al., 2019).

Consequently, the present work aims to provide a fundamental teaching framework for the completeness of quantum theory, specifically focusing on the topics of the probabilistic structure of quantum theory, EPR paradox, Bell's inequality, CHSH inequality, testing quantum theory via entanglement, quantum entanglement of bipartite systems, and quantum entanglement of spins.

2. Method

2.1. Teaching Steps

One of the most fundamental features of quantum theory is apparently the probabilistic structure, which has been continuously questioned, leading to the view that the theory is incomplete. Therefore, it is appropriate to start teaching activities from a probabilistic structure. The probabilistic structure raised the question of the completeness of the theory, and consequently, in 1935, the famous EPR paradox was expressed (Einstein et al, 1935). Some 30 years later, Bell tackled the paradox and offered an inequality to test the EPR paradox and thus the completeness of the theory (Bell, 1964). The Bell's inequality was later improved, and a simpler and more practical CHSH inequality was discovered (Clauser et al., 1969). Finally, the CHSH inequality was tested by means of quantum entanglement phenomena and, more specifically, by means of the entanglement of bipartite systems, namely, quantum spins. Subsequently, the proposed teaching framework comprises the following teaching steps;

a. Probabilistic structure of quantum theory

b. EPR paradox

c. Bell's inequality

d. CHSH inequality

e. Testing quantum theory via entanglement

f. Quantum entanglement of bipartite systems

g. Quantum entanglement of spins

The teaching approach proposed in this work assumes that students are familiar with bra-ket notation and the matrix mechanics of quantum theory. This approach is based on the well-known constructivist learning theory, which simply emphasizes that learners actively construct their own knowledge and understanding (Hein, 1991). his part should contain sufficient detail that would enable all procedures to be repeated. It can be divided into subsections if several methods are described. Authors should be as concise as possible in experimental descriptions. The experimental section must contain all of the information necessary to guarantee reproducibility. Previously published methods should be indicated by a reference and only relevant modifications should be described. For statistical analysis, please state the appropriate test(s) in addition to a hypothesized p-value or significant level (for example 0.05).

2.2. Teaching Framework

2.2.1. Step 1. Probabilistic Structure of Quantum Theory

Quantum theory resolves the behavior of matter at the atomic and subatomic scales by means of the Schrödinger wave equation, $\widehat{H}\,|\psi\rangle = E|\psi\rangle$, or in the case of quantization, it is expressed by, $\widehat{H}\,|\psi_n\rangle = E_n|\psi_n\rangle$. The relation between the overall state function $|\psi\rangle$ and the quantized state functions $|\psi_n\rangle$ is given by super position principle, $|\psi\rangle = \sum_n c_n|\psi_n\rangle$. The general formalism of standard theory cannot deterministically predict the instantaneous position of a particle or the instantaneous value of a dynamic variable. Instead, the theory can only offer probabilistic predictions, which are summarized below.

a. The theory cannot predict the exact position of a particle; instead, it predicts the existence probability function of the particle. The existence probability density is given by, $P = |\psi\rangle\langle\psi|$.

b. The theory cannot predict the instantaneous quantum state of a particle or system; rather, it predicts the probability function of a quantum state. The probability of a quantum state n is given by, $P_n = |c_n|^2 = |\langle\psi_n|\psi\rangle|^2$.

c. The theory cannot predict the instanton outcome of a specific measurement on a dynamic variable such as momentum, energy, or velocity. It can only predict the mean/expected value of a dynamic variable/operator. The expected value of an operator $\widehat{O}$ is given by, $\langle\widehat{O}\rangle = \langle\psi|\widehat{O}|\psi\rangle$.

Based on this standard formalism, the following questions should be underlined and asked the students at this stage of the teaching sequence.

a. Does the probabilistic structure represent physical reality, or is it only the result of the proposed foundational approach?

b. Does the state/wave function represent physical reality, or is it only an ontic function of the physical reality?

c. Concerning the probabilistic structure of the theory, could quantum theory be incomplete and based on an unknown hidden variable?

Consequently, quantum theory has been severely questioned by many founders regarding its probabilistic structure and indeed other absurd features, and it was then widely believed that the theory could be incomplete and needs to be improved.

2.2.2. Step 2. EPR Paradox

The probabilistic structure of quantum theory disturbed majority of the physicists however perhaps most pronouncedly Einstein and Bohr. A work on the completeness of quantum theory was first published by Einstein Podolsky and Rosen and very shortly after by Bohr in 1935 (Einstein et al., 1935; Bohr, 1935). The papers were both published in the journal of Physical Review Letters and, interestingly, they were all under the same title, specifically "Can quantum mechanical descriptions of physical reality be considered complete?" The main idea was the Einstein–Podolsky–Rosen (EPR) Paradox. The EPR paradox briefly articulates that there must be some interaction between the two dynamic variables if they are dependent on each other, in other words, if the dynamic variables are non-commuting. Consequently, if the two dynamic variables are dependent on each (non-commute), then there are two options;

a. Either way, quantum theory is not complete and there is an unknown hidden variable; consequently, the wave function cannot represent physical reality and is only an ontic function.

b. Or two dynamic dependent variables cannot simultaneously exist as a physical reality; hence, measuring one variable simultaneously vanishes the others.

It is legitimate to express that the first point of the EPR paradox has been repeatedly proved to be incorrect both theoretically and experimentally; therefore, the quantum theory appears to be complete and perfect. The second point is proved to be correct and is known as the Heisenberg uncertainty principle.

2.2.3. Step 3. Bell's Inequality

The first point of the EPR paradox, which expresses that quantum theory is not complete and the wave function cannot represent physical reality, was left completely out of interest until John Bell focused on the subject and published a pioneering paper entitled On the Einstein-Podolsky-Rosen Paradox (Bell, 1964). In his exceptional work, Bell proposed a method for testing the first point of the EPR paradox and therefore the completeness of quantum theory (Bell, 1964). The Bell testing method investigates the completeness of quantum theory by measuring the joint probabilities of instantaneous measurements carried out at three different sides: A, B, and C. Bell's theorem states that the Bell inequality, which is Equation 1.

$$|1 + P(B, C)| \geq |P(A, B) - P(A, C)| \tag{1}$$

Must be obeyed by any local hidden variable theory but should be violated by quantum mechanics. In this equation, $P(A, B)$ denotes the joint probability of the measurements of A and B, $P(B, C)$ denotes the joint probability of B and C and finally $P(A, C)$ denotes the joint probability of A and C. If Bell's inequality is experimentally verified, then one can say that quantum theory is incomplete and that an unknown hidden variable exists. It is appropriate to express at this point that Bell's inequality is violated by many experimental works that conclude the completeness of quantum theory (Aspect et al., 1981; Aspect et al., 1982a; Aspect et al., 1982b; Brunner et.al., 2014).

2.2.4. Step 4. CHSH Inequality

The Bell's inequality based on joint probability functions is later reconsidered and recalculated by a number of studies concluding different versions of the inequality; however, the most credible version is called the CHSH inequality. The CHSH inequality can simply be derived by the following approach (Clauser, Horne, Shimony and Holt, 1969).

The standard probability theory states that the joint probability of two independent events A and B is given by multiplying the independent probabilities of the two sides, that is $P(A, B) = P(A)\,P(B)$. Probability theory postulates that the probability of any event in fact originates from an unknown hidden variable, and the joint probability is determined by the hidden variable of λ. The hidden variable λ is assumed to be drawn from a fixed distribution of possible states of the source, where the probability of the source being in the state λ for any particular trial is given by the density probability function, ρ (λ), the integral of which over the complete hidden variable space is 1, $\int \rho(\lambda) d\,\lambda = 1$. Considering that the independent variable on side a is the angle α and at side b is the angle β then the correlation coefficient is defined as the expectation value of the joint probability, $C(\alpha, \beta) = \langle P(\alpha, \beta) \rangle$, which is given by Equation 2.

$$C(\alpha, \beta) = \int P(\alpha, \lambda)\, P(\beta, \lambda) \rho(\lambda) d\lambda \tag{2}$$

Where, $P(\alpha, \lambda)$ and $P(\beta, \lambda)$ are the probability functions of events A and B, respectively. Then, if α, α' are alternative angle settings for the detector at side a and β, β' for the side b, then Equation 3.

$$C(\alpha, \beta) - C(\alpha, \beta') = \int [P(\alpha, \lambda) P(\beta, \lambda) - P(\alpha, \lambda) P(\beta', \lambda)]\, \rho(\lambda) d\lambda \tag{3}$$

can be written simply. This expression can be equivalently written as Equation 4.

$$C(\alpha,\beta) - C(\alpha,\beta') = \int \big[P(\alpha,\lambda)P(\beta,\lambda)[1 \pm P(\alpha',\lambda)P(\beta',\lambda)] - P(\alpha,\lambda)P(\beta',\lambda)[1 \pm P(\alpha',\lambda)P(\beta,\lambda)]\big]\,\rho(\lambda)d\lambda \quad (4)$$

It is also obvious that the two probability functions must verify the conditions of, $|P(\alpha,\lambda)| \leq 1$ and $|P(\beta,\lambda)| \leq 1$. It can likewise be written that $[1 \pm P(\alpha',\lambda)P(\beta',\lambda)\rho(\lambda)] \geq 0$ and $[1 \pm P(\alpha',\lambda)P(\beta,\lambda)\rho(\lambda)] \geq 0$. In this case, one can easily obtain the following expression in Equation 5.

$$|C(\alpha,\beta) - C(\alpha,\beta')\,| \leq \int [1 \pm P(\alpha',\lambda)P(\beta',\lambda)]\rho(\lambda)d\lambda - \int [1 \pm P(\alpha',\lambda)P(\beta,\lambda)]\,\rho(\lambda)d\lambda \quad (5)$$

This inequality can alternatively be stated in terms of the definition given in the Equation 6:

$$|C(\alpha,\beta) - C(\alpha,\beta')\,| \leq 2 \pm |C(\alpha',\beta') + C(\alpha',\beta)\,| \quad (6)$$

In fact, this formulation expresses the CHSH inequality in terms of correlation coefficients, and it can be written in its more familiar final form as follows Equation 7.

$$|S| = |C(\alpha,\beta) - C(\alpha,\beta') + C(\alpha',\beta') + C(\alpha',\beta)\,| \leq 2 \quad (7)$$

In this inequality, S represents the Clauser, Horne, Shimony, and Holt (CHSH) *correlation parameters*. Consequently, regarding a series of measurements or calculations, two options are possible;

a. If the correlation parameter is equal or less than 2 ($|S| \leq 2$) then it leads to the result that quantum theory is *incomplete* and there is a hidden variable which is not known.

b. If the correlation parameter is greater than 2 ($|S| > 2$) then it leads to the result that quantum theory is *complete* and perfect.

As a result, it is obvious that the CHSH inequality provides a useful tool to test the completeness of quantum theory both theoretically and experimentally.

2.2.5. Step 5: Quantum Theory Testing via Entanglement

Schrödinger originally proposed quantum entanglement in a pioneering work entitled *Discussion of probability relations between separated systems* (Schrödinger, 1935). The fundamental point is that quantum systems can interact through their state/wave functions even though they are spatially distant from each other. Hence, this phenomenon, if possible, would verify the nonlocality of quantum theory, which was named by Einstein's spooky action at a distance. However, the entanglement of quantum particles depends on the background of the particles in the sense that the particles should be coming from the same origin, and the state/wave functions should overlap once in their history. Hence, it is not possible to observe quantum entanglement for randomly chosen two or more quantum particles. This condition is, for instance, realized by removing two electrons from a single atom or two or more photons from the same semiconductor laser. Entanglement means that the quantum states of two particles are instantly coupled, in other words, influencing each other. Therefore, the measurement of the first quantum state instantly determines the quantum state of the second particle. The interaction of two states, for instance A and B, can be expressed by a joint function, P (A, B), as expressed within Bell's and CHSH inequalities.

The CHSH inequality is, on the other hand, obviously an applicable tool to test the EPR paradox and consequently Bell's inequality and completeness of quantum theory via the quantum entanglement phenomenon. The CHSH inequality is derived based on two fundamental features:

a. A hidden variable connects the two measurement systems.

b. The hidden variable maintains locality.

if and when the inequality is verified, then it means that there is a hidden variable, quantum theory is incomplete, and quantum theory maintains locality. To test the CHSH inequality, two separate quantum systems far apart from each other (A and B) and two separate measurement cases or directions (α, α' for A and $\beta, \beta'$for B) are required. On the other hand, to measure the correlation coefficients for the two quantum particles under consideration, they should be coupled or entangled via state/wave functions even though the particles are distant from each other. . If, on the other hand, the CHSH inequality is violated through quantum entanglement, then the quantum theory is complete and maintains non-locality. The experimental realization of the CHSH inequality was tested many times via quantum entanglement but most pronouncedly realized by Aspect and concluded that the CHSH inequality is violated; thus, quantum theory is complete, and quantum theory maintains nonlocality (Aspect et al., 1981; Aspect et al., 1982a; Aspect et al., 1982b).

2.2.6. Step 6: Quantum Entanglement of Bipartite Systems

The fundamental theory of quantum mechanics describes the behavior of any quantum mechanical system or particle using state/wave functions, which are obtained from the solution of Schrödinger's wave equation. Assuming that the particle is spatially confined, then quantization occurs, and any pure state can be expressed by, $|\psi_n\rangle$, and the fundamental theory also describes the overall/mixed state function $|\psi\rangle$ as the linear combination of the pure state functions through the well-known superposition principle in Equation 8.

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle \tag{8}$$

where c_n represents the coefficient for the n. quantum state, and the probability of n. quantum state is defined by, $P_n = |c_n|^2$.

If one assumes bipartite systems, which means that only two quantized states are possible like spin up and spin down for the spin state function, then the overall state function can be expressed as Equation 9.

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \tag{9}$$

If two separate quantum systems, namely systems A and B with bipartite features, are present and combined, then the mixed state functions for systems A and B can be given by the following Equation 10 and Equation 11.

$$|\psi_A\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle \tag{10}$$

$$|\varphi_B\rangle = b_1|\varphi_1\rangle + b_2|\varphi_2\rangle \tag{11}$$

In these equations, the coefficients are solidly defined by standard quantum mechanics as the scalar products of the specific state functions and the overall mixed state functions in Equation 12, Equation 13, Equation 14, and Equation 15:

$$a_1 = \langle\psi_1|\psi_A\rangle \tag{12}$$

$$a_2 = \langle\psi_2|\psi_A\rangle \tag{13}$$

$$b_1 = \langle\varphi_1|\varphi_B\rangle \tag{14}$$

$$b_2 = \langle\varphi_2|\varphi_B\rangle \tag{15}$$

The probabilities of quantum states for systems A and B are defined by the squared amplitudes of the relevant coefficients in the following form, $P_1^A = |a_1|^2$, $P_2^A = |a_2|^2$, $P_1^B = |b_1|^2$ and $P_2^B = |b_2|^2$. Therefore, individual probabilities can be expressed as follows in Equation 16, Equation 17, and Equation 18:

$$P_1^A = |\langle\psi_1|\psi_A\rangle|^2 \qquad 16$$

$$P_2^A = |\langle\psi_2|\psi_A\rangle|^2 \qquad 17$$

$$P_1^B = |\langle\varphi_1|\varphi_B\rangle|^2 \qquad 18$$

$$P_2^B = |\langle \varphi_2 | \varphi_B \rangle|^2 \quad 19$$

Considering two separate bipartite quantum systems or particles of A and B, the state function of the composite system is given by the tensor product of the two individual state functions in Equation 20:

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\varphi_B\rangle \quad 20$$

The composite system can have two different features. The first state can be called a separable state in which the two systems are completely independent of each other and the state functions do not interact with each other. The second feature involves the interaction of the two systems; therefore, the state functions are not independent of each other. This feature is quantum mechanically possible, and it is referred to as an entangled state. The two cases can be resolved as follows.

Separable States

In this case, the quantum particles at the two sides can be completely independent of each other, and the state functions do not interact by any means. Hence, by substituting the definitions given by Equation10 and 11, the composite state function can be written in terms of the individual state functions of systems A and B as follows in Equation 21:

$$|\psi_{AB}\rangle = a_1 b_1 |\psi_1\rangle|\varphi_1\rangle + a_1 b_2 |\psi_1\rangle|\varphi_2\rangle + a_2 b_1 |\psi_2\rangle|\varphi_1\rangle + a_2 b_2 |\psi_2\rangle|\varphi_2\rangle \quad 21$$

The joint probability functions of the two independent sides are expressed by,$P_{11} = |a_1\, b_1|^2 = |a_1\,|^2 |b_1|^2 = P_1^A\, P_1^B$, $P_{12} = |a_1\, b_2|^2 = |a_1\,|^2 |b_2|^2 = P_1^A\, P_2^B$, $P_{21} = |a_2\, b_1|^2 = |a_2\,|^2 |b_1|^2 = P_2^A\, P_1^B$ and finally $P_{22} = |a_2\, b_2|^2 = |a_2\,|^2 |b_2|^2 = P_2^A\, P_2^B$.

Accordingly, the joint probability functions for the separable states can be formulated as follows in Equation 22, Equation 23, Equation 24, and Equation 25.

$$P_{11} = P_1^A\, P_1^B \quad (22)$$

$$P_{12} = P_1^A\, P_2^B \quad (23)$$

$$P_{21} = P_2^A\, P_1^B \quad (24)$$

$$P_{22} = P_2^A\, P_2^B \quad (25)$$

It is clear that if the states are separable, then the classical joint probabilities are given by the products of the individual probability functions, as in the classical cases.

Entangled States

If, on the other hand, the two quantum particles/systems come from the same origin, then the quantum particles at the two sides cannot be independent of each other and can therefore interact with each other. This case is called entanglement of quantum particles. In this case, the state functions are entangled, and the state function of the composite system can be Equation 26.

$$|\psi_{AB}\rangle = c_{11} |\psi_1\rangle|\varphi_1\rangle + c_{12} |\psi_1\rangle|\varphi_2\rangle + c_{21} |\psi_2\rangle|\varphi_1\rangle + c_{22} |\psi_2\rangle|\varphi_2\rangle \quad (26)$$

In this formulation, the coefficients must be defined for the entangled systems and have the same physical meaning as those previously explained. The joint probability functions can similarly be formulated in terms of the coefficients in the following arrangement in Equation 27, Equation 28, Equation 29, and Equation 30:

$$P_{11} = |c_{11}|^2 \quad (27)$$

$$P_{12} = |c_{12}|^2 \quad (28)$$

$$P_{21} = |c_{21}|^2 \tag{29}$$

$$P_{11} = |c_{22}|^2 \tag{30}$$

In this case, the coefficients for the entangled quantum states are completely different and should be defined carefully. In order to define the entangled coefficients, one can theoretically multiply the composite system wave function, $|\psi_{AB}\rangle$, (equation 26) from the left-hand side by the two bra wave functions of the first quantum system, that is $\langle\psi_1|\,\langle\psi_1| = \langle\psi_1\psi_1| = \langle\psi_{11}|$. Then one can straightforwardly obtain the following Equation 31:

$$\langle\psi_1|\,\langle\psi_1|\;|\psi_{AB}\rangle = c_{11}\,\langle\psi_1|\,\langle\psi_1|\;|\psi_1\rangle|\varphi_1\rangle + c_{12}\langle\psi_1|\,\langle\psi_1|\;|\psi_1\rangle|\varphi_2\rangle + c_{21}\langle\psi_1|\,\langle\psi_1|\,|\psi_2\rangle|\varphi_1\rangle + c_{22}\langle\psi_1|\,\langle\psi_1|\;|\psi_2\rangle|\varphi_2\rangle \tag{31}$$

In this equation, it is obvious that the orthonormality of the Hilbert space holds, and one can write in Equation 32, and Equation 33.

$$\langle\psi_1|\psi_1\rangle = 1 \tag{32}$$

$$\langle\psi_1|\psi_2\rangle = 0 \tag{33}$$

Then the Equation 31 can be written as follows in Equation 34:

$$\langle\psi_1|\,\langle\psi_1|\;|\psi_{AB}\rangle = c_{11}\langle\psi_1|\varphi_1\rangle + c_{12}\langle\psi_1|\varphi_2\rangle \tag{34}$$

This equation demonstrates that only state 1 in system A can couple with states 1 and 2 in system B. The overall probability function for this specific case can be written as follows in Equation 35:

$$\langle\psi_{11}|\psi_{AB}\rangle = |c_{11}|^2 + |c_{12}|^2 \tag{35}$$

Accordingly, the coefficients can be easily formulated as Equation 36.

$$c_{11} = \langle\psi_1|\varphi_1\rangle \tag{36}$$

where is the coefficient of the entanglement of $\psi_1$and φ_1, and Equation 37 is

$$c_{12} = \langle\psi_1|\varphi_2\rangle \tag{37}$$

where is the coefficient for the entanglement of $\psi_1$and φ_2.

The same procedure can be applied to obtain the other coefficients, namely c_{21} and c_{22}. If the composite system wave function, $|\psi_{AB}\rangle$ this time by the second bra state functions of the quantum system A, that is $\langle\psi_2|\,\langle\psi_2| = \langle\psi_2\psi_2| = \langle\psi_{22}|$. Then, one can obtain the following Equation 38:

$$\langle\psi_2|\,\langle\psi_2|\;|\psi_{AB}\rangle = c_{11}\,\langle\psi_2|\,\langle\psi_2|\;|\psi_1\rangle|\varphi_1\rangle + c_{12}\langle\psi_2|\,\langle\psi_2|\;|\psi_1\rangle|\varphi_2\rangle + c_{21}\langle\psi_2|\,\langle\psi_2|\,|\psi_2\rangle|\varphi_1\rangle + c_{22}\langle\psi_2|\,\langle\psi_2|\;|\psi_2\rangle|\varphi_2\rangle \tag{38}$$

Likewise, in this expression, the orthogonality and normalization conditions must be verified in Equation 39 and Equation 40.

$$\langle\psi_2|\psi_2\rangle = 1 \tag{39}$$

$$\langle\psi_2|\psi_1\rangle = 0 \tag{40}$$

The Equation 38 is then reduced to the following Equation 41:

$$\langle\psi_{22}|\psi_{AB}\rangle = c_{21}\langle\psi_2|\varphi_1\rangle + c_{22}\langle\psi_2|\varphi_2\rangle \tag{41}$$

The total probability for this case can similarly be expressed by, $\langle\psi_{22}|\psi_{AB}\rangle = |c_{21}|^2 + |c_{22}|^2$, and it leads to the definitions of equation 42 and Equation 43,

$$c_{21} = \langle\psi_2|\varphi_1\rangle \quad (42)$$

$$c_{22} = \langle\psi_2|\varphi_2\rangle \quad (43)$$

It is now obvious that coefficient (42) denotes the coefficient for the entanglement of $\psi_2$and φ_1 and the coefficient (43) defines the entanglement of ψ_2 and φ_2. The joint probability functions of the entangled states are accordingly given by Equation 44, Equation 45, Equation 46, and Equation 46.

$$P_{11} = |\langle\psi_1|\varphi_1\rangle|^2 \quad (44)$$

$$P_{12} = |\langle\psi_1|\varphi_2\rangle|^2 \quad (45)$$

$$P_{21} = |\langle\psi_2|\varphi_1\rangle|^2 \quad (46)$$

$$P_{11} = |\langle\psi_2|\varphi_2\rangle|^2 \quad (47)$$

These theoretical resolutions for the entangled states can now be combined with the equations of the CHSH inequality. The joint probability for sides A and B was defined as $\langle P(A,B)\rangle = C(A,B)$ which is obviously equal to the joint probability of the composite system. Specifically, the entanglement correlation coefficient can be written in terms of the joint probability functions as Equation 48.

$$C(A,B) = P_{11} + P_{22} - P_{12} - P_{21} \quad (48)$$

Finally, the entanglement parameter in the CHSH inequality can be defined as follows Equation 49:

$$|S| = |C(A,B) - C(A,B') + C(A',B') + C(A',B)\ | \quad (49)$$

In this expression A and A'represent two different measurement directions for the first particle/system and similarly B and B'represent two different measurement directions for the second particle/system.

2.2.7. Step 7. Quantum Entanglement of Spins

The theoretical resolution of entanglement outlined previously can be theoretically applied to the spin entanglement of quantum particles because the spin features of quantum particles are the best examples of bipartite systems. Any particle like electron, proton, neutron, or photon, has a spin feature because it rotates around its central axis. If the rotation is in the direction of anticlockwise, then it is called spin up, and if it is in the direction of clockwise, then it is called spin down. The spin features of quantum particles are best resolved in terms of spherical coordinates, shown in Figure 1.

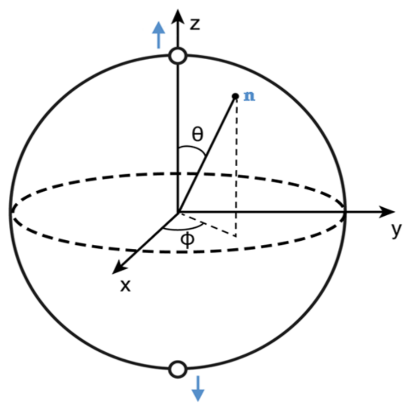

Figure 1. The figure shows a randomly chosen direction of spin $\vec{n}$ having angles of $\theta$and ϕ within the Bloch sphere with a radius of unity, $|\vec{n}| = n = 1$.

In order to resolve the spin state vectors and coefficients, the randomly chosen direction of the spin vector can be represented as Equation 50

$$\vec{n} = n_x\vec{i} + n_y\vec{j} + n_z\vec{k} \tag{50}$$

With the magnitude of the vector; therefore, the radius of the sphere is equal to unity, $|\vec{n}| = n = 1$. The components of the unit vector $\vec{n}$ can obviously be expressed by Equation 51, Equation 52, and Equation 53.

$$n_x = \sin\theta\cos\phi \tag{51}$$

$$n_y = \sin\theta\sin\phi \tag{52}$$

$$n_z = \cos\theta \tag{53}$$

The scalar product of the unit vectors can be written by, $\vec{n}\,.\,\vec{n} = n^2 = n_x{}^2 + n_y{}^2 + n_z{}^2$ and it is also equal to unity 1 (Equation 54).

$$\vec{n}\,.\,\vec{n} = 1 \tag{54}$$

What we are looking for is a general spin operator that can be employed for any randomly chosen direction with an absolute magnitude of the unity, 1. Accordingly, it is appropriate to define the spin operator as the scalar products of the unit vectors in the up and down orientations for the same direction. The spin operator can be formulated as Equation 55.

$$\hat{S} = -\vec{n}\,.\,\vec{n} \tag{55}$$

Please note that this fundamental definition includes both the spin-up and spin-down components of the concerning direction and has an absolute magnitude of 1. The equivalent matrix form of the spin operator can now be expressed as a 2x2 matrix by Equation 56:

$$\hat{S} = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix} \tag{56}$$

whose determinant is equal to $|\hat{S}| = -(n_x{}^2 + n_y{}^2 + n_z{}^2) = -1.$ Substitution of the components given by the Equations 51, 52 and 53, leads to the conclusion that Equation 57.

$$\hat{S} = \begin{pmatrix} \cos\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & -\cos\theta \end{pmatrix} \tag{57}$$

Specific components of the spin matrix operator can be obtained by substituting relevant angles in this general expression. Specifically, the x component of the spin matrix operator is given by substituting $\theta = \frac{\pi}{2}$ and $\phi = 0$, y component is obtained by $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$ and z component is obtained by $\theta = 0$ and ϕ =anything. Accordingly, the spin matrix elements called Pauli spin matrices can be expressed as follows Equation 58, Equation 59, and Equation 60:

$$\hat{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{58}$$

$$\hat{S}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ i & 0 \end{pmatrix} \tag{59}$$

$$\hat{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{60}$$

The standard formalism of quantum mechanics can now be applied to the spin matrix operator. The spin, as stated previously, is a bipartite quantum system with two options: spin up, $|\psi_\uparrow\rangle$ and spin down, $|\psi_\downarrow\rangle$. The well-known superposition principle is employed to express the overall state function of the mixed state Equation 61:

$$|\psi\rangle = c_\uparrow|\psi_\uparrow\rangle + c_\downarrow|\psi_\downarrow\rangle \tag{61}$$

where $c_\uparrow$ and $c_\downarrow$ denote the relevant coefficients of the spin up and down cases, respectively. To calculate the entanglement of quantum spins, specific wave functions and coefficients should be specified. The well-known Eigen function-Eigen value equation can be written for the spin up case as follows in Equation 62:

$$\hat{S}\,|\psi_\uparrow\rangle = +1\,|\psi_\uparrow\rangle \tag{62}$$

Substituting the relevant expressions with $|\psi_\uparrow\rangle = \begin{pmatrix}\gamma\\ \delta\end{pmatrix}$ where γ and δ represents the component of 2x1 spin up matrix element, gives $\begin{pmatrix}\cos\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = +1\begin{pmatrix}\gamma\\ \delta\end{pmatrix}$ which clearly leads to the conclusion that Equation 63.

$$|\psi_\uparrow\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ \mathrm{e}^{\mathrm{i}\phi}\sin\frac{\theta}{2}\end{pmatrix} \tag{63}$$

The same procedure can be applied to the spin-down case can be seen in Equation 64.

$$\hat{S}\,|\psi_\downarrow\rangle = -1\,|\psi_\downarrow\rangle \tag{64}$$

and the Eigen value-Eigen function equation can be written as $\begin{pmatrix}\cos\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = -1\begin{pmatrix}\gamma\\ \delta\end{pmatrix}$, which gives the spin-down state in Equation 65.

$$|\psi_\downarrow\rangle = \begin{pmatrix}\sin\frac{\theta}{2}\\ -\mathrm{e}^{\mathrm{i}\phi}\cos\frac{\theta}{2}\end{pmatrix} \tag{65}$$

In order to calculate the coefficients, it is appropriate to employ the superposition principle, which was previously given by the Equation 9, $|\psi\rangle = c_\uparrow|\psi_\uparrow\rangle + c_\downarrow|\psi_\downarrow\rangle$. If one multiplies this expression from the left-hand side by the bra spin up state vector in the z direction, $\langle\psi_\uparrow^z|$, one obtains can be seen in Equation 66

$$\langle\psi_\uparrow^z|\psi\rangle = c_\uparrow\langle\psi_\uparrow^z|\psi_\uparrow\rangle + c_\downarrow\langle\psi_\uparrow^z|\psi_\downarrow\rangle \tag{66}$$

The overall probability of spin up in the z direction is defined as follows in Equation 67:

$$|\langle\psi_\uparrow^z|\psi\rangle|^2 = |c_\uparrow|^2 + |c_\downarrow|^2 \tag{67}$$

Hence, it is now straightforward to define specific coefficients as follows by Equation 68 and Equation 69:

$$c_\uparrow = \langle\psi_\uparrow^z|\psi_\uparrow\rangle \tag{68}$$

$$c_\downarrow = \langle\psi_\uparrow^z|\psi_\downarrow\rangle \tag{69}$$

These definitions can now be employed because we know the state function of $|\psi_\uparrow^z\rangle$. The general formulation of the spin up state function was previously derived and given by equation (63). If one substitutes specific values for the z direction which are $\theta = 0$ and ϕ= anything then it is clear that the spin up z state function is equal to, $|\psi_\uparrow^z\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$. It is now easy to calculate the specific coefficients,

$c_{\uparrow} = \langle\psi_{\uparrow}^{z}|\psi_{\uparrow}\rangle = (1 \quad 0)\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = \cos\frac{\theta}{2}$ and $c_{\downarrow} = \langle\psi_{\uparrow}^{z}|\psi_{\downarrow}\rangle = (1 \quad 0)\begin{pmatrix}\sin\frac{\theta}{2}\\ -e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix} = \sin\frac{\theta}{2}$. As a result, the spin up and spin down coefficients are respectively derived as Equation 70 and Equation 71.

$$c_{\uparrow} = \cos\frac{\theta}{2} \tag{70}$$

$$c_{\downarrow} = \sin\frac{\theta}{2} \tag{71}$$

In order to observe the spin entanglement of the two particles, one obviously needs two separate quantum particles, one at side A and the other at side B, with the same origin so that the state functions are entangled. In this case, let α, α' being alternative angle settings for the detector at side A and β, β' for the side B. Considering specifically the measurements at the angle α at A and β at B, we find that the joint probability coefficient for the spin up at both sides can be found as, $c_{\uparrow\uparrow} = \left\langle\psi_{\uparrow}^{\alpha}|\psi_{\uparrow}^{\beta}\right\rangle = \left(\cos\frac{\alpha}{2} \quad e^{-i\phi}\sin\frac{\alpha}{2}\right)\begin{pmatrix}\cos\frac{\beta}{2}\\ e^{i\phi}\sin\frac{\beta}{2}\end{pmatrix} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}$ which leads to the conclusion that Equation 72.

$$c_{\uparrow\uparrow} = \cos\frac{\alpha-\beta}{2} \tag{72}$$

Hence, it is now straightforward to calculate the joint probability of having a spin up at A and a spin up at B, $P_{\uparrow\uparrow} = |c_{\uparrow\uparrow}|^2$, can be seen in Equation 73.

$$P_{\uparrow\uparrow} = \left|\cos\frac{\alpha-\beta}{2}\right|^2 \tag{73}$$

The same calculation can be performed for the case of spin down-spin down case and the same result can be obtained for the joint probability, $P_{\downarrow\downarrow} = |c_{\downarrow\downarrow}|^2$, can be seen in Equation 74.

$$P_{\downarrow\downarrow} = \left|\cos\frac{\alpha-\beta}{2}\right|^2 \tag{74}$$

The definition derived above can be employed for the case of spin up for side A and spin down for side B. The coefficient for the spin up-spin down case can be calculated as follows: $c_{\uparrow\downarrow} = \left\langle\psi_{\uparrow}^{\alpha}|\psi_{\downarrow}^{\beta}\right\rangle = \left(\cos\frac{\alpha}{2} \quad e^{-i\phi}\sin\frac{\alpha}{2}\right)\begin{pmatrix}\sin\frac{\beta}{2}\\ -e^{i\phi}\cos\frac{\beta}{2}\end{pmatrix} = \cos\frac{\alpha}{2}\sin\frac{\beta}{2} - \sin\frac{\alpha}{2}\cos\frac{\beta}{2}$ and the coefficient is then found as Equation 75.

$$c_{\uparrow\downarrow} = \sin\frac{\alpha-\beta}{2} \tag{75}$$

The joint probability of spin up-spin down can be calculated by, $P_{\uparrow\downarrow} = |c_{\uparrow\downarrow}|^2$, then can be seen in Equation 76.

$$P_{\uparrow\downarrow} = \left|\sin\frac{\alpha-\beta}{2}\right|^2 \tag{76}$$

The same procedure can be carried out for the case of spin down-spin up and the joint probability can be derived as, $P_{\downarrow\uparrow} = |c_{\downarrow\uparrow}|^2$, then can be seen in Equation 77.

$$P_{\downarrow\uparrow} = \left|\sin\frac{\alpha-\beta}{2}\right|^2 \tag{77}$$

To determine the completeness of the quantum theory, the correlation coefficient within the CHSH inequality can be calculated as Equation 78.

$$C(\alpha,\beta) = P_{\uparrow\uparrow} + P_{\downarrow\downarrow} - P_{\uparrow\downarrow} - P_{\downarrow\uparrow} = 2\ \left|\cos\frac{\alpha-\beta}{2}\right|^2 - 2\left|\sin\frac{\alpha-\beta}{2}\right|^2 \quad (78)$$

which leads to the final expression of Equation 79.

$$C(\alpha,\beta) = 2\cos(\ \alpha - \beta) \quad (79)$$

The same procedure can be applied to the other correlation coefficients, namely $C(\alpha,\beta'), C(\alpha',\beta'), C(\alpha',\beta)$ and CHSH entanglement correlation parameter can be calculated as follows Equation 80:

$$|S| = |C(\alpha,\beta) - C(\alpha,\beta') + C(\alpha',\beta') + C(\alpha',\beta)\ | \quad (80)$$

which leads to the final expression of Equation 81.

$$|S| = 2[\cos(\alpha-\beta) - \cos(\alpha-\beta') + \cos(\alpha'-\beta') + \cos(\alpha'-\beta)] \quad (81)$$

This equation gives the overall *entanglement correlation parameter* for the spin measurements at four different measurement angles.

One can now test the validity of the CHSH inequality and therefore the completeness of quantum theory by randomly selecting angles and calculating the $|S|$ entanglement parameters. This is managed and tabulated in the Table 1.

Table 1. The entanglement parameters were theoretically calculated using Equation 81 for five different angle combinations, all of which demonstrate the completeness of quantum theory.

Case	$\alpha(Deg)$	$\beta(Deg)$	$\alpha'(Deg)$	$\beta'(Deg)$	$\|S\|$
1	45	22.5	0	-22.5	4.1268
2	60	30	15	-15	4.8782
3	10	60	30	-30	2.4857
4	30	20	45	-20	3.1676
5	90	70	-40	10	2.1337

The calculated values for the entanglement parameter verify the condition of $|S| > 2$), which expresses the completeness and non-locality of quantum theory.

his part should contain sufficient detail that would enable all procedures to be repeated. It can be divided into subsections if several methods are described. Authors should be as concise as possible in experimental descriptions. The experimental section must contain all of the information necessary to guarantee reproducibility. Previously published methods should be indicated by a reference and only relevant modifications should be described. For statistical analysis, please state the appropriate test(s) in addition to a hypothesized p-value or significant level (for example 0.05).

3. Results and Discussion

The development of teaching material for any course in accordance with novel scientific and technological advances is vital. In this regard, quantum theory stands at the forefront of the field and should be handled. To evaluate the teaching implications, the teaching proposal was presented to a total of 35 undergraduate quantum mechanics students and was voluntarily asked to answer the following questions:

a. What is your opinion on the necessity of imparting complete teaching of quantum theory and quantum entanglement?

b. What is your opinion about the simplicity of teaching the completeness of quantum theory and quantum entanglement?

c. What are your opinions on the significance of the complete teaching of quantum theory and quantum entanglement?

d. What are your opinions on the importance of the complete teaching of quantum theory and quantum entanglement?

e. What is your opinion about the originality of the teaching completeness of quantum theory and quantum entanglement?

The students are asked to evaluate their opinions and grade in terms of 5 Likert type scale, specifically comprised of very high, high, medium, low, and very low choices. The percentages of student answers are presented in **Table** 2.

Table 2. Percentage of student opinions on the teaching proposed teaching framework.

Subject	Very low (%)	Low (%)	Medium (%)	High (%)	Very high (%)
Necessity	-	-	4.0	85.6	10.4
Simplicity	-	28.5	14.3	57.2	-
Significance	-	-	57.2	22.9	19.9
Importance	-	-	68.6	17.1	14.3
Originality	-	-	2.9	91.1	6.0

This table clearly reveals that the majority of students believe that the necessity, simplicity, and originality of the teaching proposal are expected to be high and that the significance and importance are expected to be medium.

4. Conclusion

The present work proposes a teaching proposal concerning the completeness of quantum theory and the fascinating phenomenon of quantum entanglement. The framework is specifically comprised of the probabilistic structure of quantum theory, the EPR paradox, Bell's inequality, the CHSH inequality, testing quantum theory via entanglement, quantum entanglement of bipartite systems, and quantum entanglement of spins. This work offers nearly full theoretical resolution for quantum entanglement and spin entanglement. The views of the students regarding the teaching framework indicate that necessity, simplicity, and originality are high and that significance and importance are medium. The proposed teaching framework is reasonably simple and includes almost all the details that can be easily introduced to undergraduate-level quantum mechanics courses.

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