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Chaos to fractals

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Abstract: In undergraduate classrooms, while teaching chaos and fractals, it is taught as if there is no relation between these two. By using some non linear oscillators we demonstrate that there is a connection between chaos and fractals. By plotting the phase space diagrams of four nonlinear oscillators and using box counting method of finding the fractal dimension we established the chaotic nature of the nonlinear oscillators. The awareness that all chaotic systems are good fractals will add more insights to the concept of chaotic systems.
Keywords: fractals; nonlinear oscillators

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Introduction

Most of the phenomena in nature do not show regular and periodic progress over time since there is an inherent randomness in all of them (Atakan et al., 2019; Öztürk, 2020). The random behavior of a process or a system over a period of time is called chaos. Some examples for chaotic systems include weather, rainfall over a large period, ocean turbulence, smoke plumes etc. (Shukla, 1998). Chaos theory is a mathematical field of study which states that non-linear dynamical systems that are seemingly random are actually deterministic (Biswas et al., 2018). Chaos theory, or deterministic chaos, may be traced back to mathematician Henri Poincare, working at the end of the 19th century, and more recently to meteorologist Edward Lorenz (Forgues & Thietart, 2016). Chaos theory begin with Poincare's study of three body problem and was confirmed after the experiments of Lorentz, who was a meteorologist. His key observation was the sensitive dependence of initial conditions. Slight changes in the initial values may result in large errors in the final value(Motter& Campbell, 2013). Using the chaos theory, we can understand a complicated system more easily or increase the predictability of the system. Now, with the advent of modern computers chaos is studied numerically to track the time evolution of the system with the changes in initial conditions (Cattani et al., 2017).

Chaotic systems exhibit some types of interconnections, patterns and self-similarity. In undergraduate classrooms, students study the motion of a pendulum and in laboratories they find that the pendulums always exhibit periodic motion for all observations. Actually this is due to the strict restriction given by the teacher that the angular displacement must be small. But in reality, when the initial displacement is made large, the pendulums exhibit chaotic motion (Adams & Russ, 1992; Palmore, 1991). In nature many structures like mountains, clouds, leaves etc., have a structure that seem to be highly irregular in the first view, but when closely examined at different scales, repeated patterns are observed. Such structures are called fractals and self-similarity is one of the basic character of the fractals. There has been many studies connecting chaos and fractal (Boeing, 2016; Korolj et al., 2019; Zhu et al., 2021).Here in this paper we give some simple techniques to establish the relationship

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between chaos and fractals which may help the students to have a clarity of both chaos and fractals and to have an understanding of their relations.

Methods for Detecting Chaos

There are various methods for detecting chaos in a system (Özer & Akin, 2005). They are by plotting

1. Phase space diagrams
2. Time series
3. Poincare maps
4. Power spectrum
5. Lyapunov exponents
6. Bifurcation diagram.

Phase space

Phase space is a 6N dimensional space with position and momentum co-ordinates. Each state of the system is represented by a unique point in phase space. As the system changes with time, the point traverses a path in the phase space called trajectory. If the system is chaotic the trajectory will pass through almost all the regions of phase space (Beale & Pathria, 2011).

Time series

Time series analysis is used to study the dynamics and evolution of systems of any type. Its main aim is to extract information with a minimum number of parameters needed for approximating the data within a given interval (Andronov, 2020). We take a variable of the system in Y axis and time along X axis. By drawing the time series, we can check whether the motion is periodic or chaotic directly.

Poincare section

The Poincare section is the discrete set of phase space points of the system for each cycle. If T is the time taken for one cycle of motion, then after each T, Poincare section will give one point. But if the motion is periodic then the Poincare section will have only one point since the points after each T will be same. If the period is doubled it will have two points. Since chaotic motion is a period infinite motion, the Poincare section of a chaotic system will be filled by large number of points.

Power Spectrum

Chaotic signals are wideband signals. So by observing the frequency spectra we can easily distinguish chaotic signals from periodic signals. If the system is chaotic, then power spectra is expressed in terms of oscillations with a continuum of frequencies(Özer & Akin, 2005).

Lyapunov exponents

The logarithmic measure for the mean expansion rate per iteration of the distance between two infinitesimally close trajectories is the Lyapunov exponent of the system. A chaotic system will have a positive Lyapunov exponent. The paths of such systems are extremely sensitive to the initial conditions (Greiner, 2010).

Bifurcation diagram

Bifurcation diagram shows the long term changes in the behaviour a system by varying the control parameter. For some values of the parameter, the system will have only one long term motion while for some other slightly different values, system may have two or three motions. This means the behaviour of the system depends on initial condition. In differential equations, if a change in the number of solution is depending on parameter variation, it is called bifurcation (Özer & Akin, 2005).

In this article we will study and plot phase space,time series and Poincare maps, because of their simplicity in understanding. We will use four nonlinear differential equations for the study.

Linear and Nonlinear Differential Equations

In our surroundings we come across many events like motion, heat flow, wind etc. In physics, we model them using differential equations whose solutions give the description of the systems. There are mainly two types of differential equations-Linear and nonlinear differential equations. Linear means that the variable in an equation appears only with a power of one. So x is linear but x2 is non-linear.

1. In a differential equation, when the variables and their derivatives are only multiplied by constants, not functions, then the equation is linear, if the variables and their derivatives appear as a simple first power,

x'' + x = 0, the equation is linear.

x'' + 2x' + x = 0, is linear.

But x' + $\frac{1}{x}$= 0 is non-linear because $\frac{1}{x}$ is not a first power.

x' + x^2 = 0 is non-linear because x^2 is not a first power.

Similarlyx'' + $sin(x)$ = 0 is non-linear because $sin(x)$is not a first power, since on expansion $sin(x)$ = x - $\frac{x^3}{3!}$ + $\frac{x^5}{5!}$ - .....

Againxx' = 1 is non-linear because x' is not multiplied by a constant, besides it can also be written as$\frac{d}{dt}\left(\frac{x^2}{2}\right) = 1.$

The above examples clearly differentiates a linear differential equation from a nonlinear differential equation(Arfken et al., 2013; Fusic & Kufner, 2014; Struble, 2018).

Results and Discussions

Here we will consider four most used and discussed nonlinear oscillators - Duffing oscillator, HenonHeiles oscillator, Quartic oscillator, and Van der pol oscillator. In the next subsections we will have a short description of these oscillators.

Duffing oscillator

The nonlinear equation describing an oscillator with a cubic non linearity is called the Duffing equation named after Georg Duffing, a German engineer (Kovacic & Brennan, 2011). The differential equation can be written as,

$$\boldsymbol{\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma cos(\omega t)} \tag{1}$$

where the (unknown) function x = x(t) is the displacement at time t. The parameters in the Duffing equation are;δ is the amount of damping,α is the linear stiffness, β is the amount of non-linearity in the restoring force; ifβ = 0, the Duffing equation describes a damped and driven simple harmonic oscillator, γ is the amplitude of the periodic driving force; if γ= 0 the system is without a driving force and ω is the angular frequency of the periodic driving force.

Many physical systems are conveniently modeled by the Duffing equation. For example, it is used to study the solutions of sine-Gordon equation, Klein–Gordon equation, nonlinear Schrodinger equation etc. (Humberto et al., 2021). The Duffing equation basically represents the motion of a damped, driven inverted pendulum with a torsional restoring force.

The Hamiltonian of an undamped, unforced Duffing oscillator is,

$$\boldsymbol{H = \frac{1}{2m}p_x^2 + \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4} \tag{2}$$

where p_x is the momentum in x direction.

Henon Heiles oscillator

The Henon Heiles potential was developed by Michel Henon and Carl Heiles to check the possibility of a third isolating integral of motion in celestial mechanics(Henon& Heiles, 1964). The Hamiltonian of a Henon Heiles oscillator is

$$\boldsymbol{H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3 + x^2 y} \qquad (3)$$

where x, y are position and $p_x, p_y$are momentum coordinates.

Michel Henon and Carl Heiles while studying the galactic motion attempted to solve the question 'Does an axisymmetrical potential admit a third isolating integral of motion?' by numerical computations (Henon & Heiles, 1964). They showed that this potential is equivalent to the problem of the motion a particle in a plane in an arbitrary potential U. After several trials they had taken the following potential for study

$$\boldsymbol{U(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3 + x^2 y} \qquad (4)$$

because it is analytically simple, this makes the computation of the trajectory easy and at the same time, it is sufficiently complicated to give trajectories which are far from trivial. Using this potential they have done several numerical computations and concluded that if the energy is small, it seems that a third isolating integral always exists (Henon & Heiles, 1964). For energy greater than$\frac{1}{6}$the oscillator is completely chaotic. By simply increasing the energy we can see the transition from an integrable system to a chaotic system.

Quartic Oscillator

The differential equation for a pure quartic oscillator is

$$m\frac{d^2x}{dt^2} + Kx^3 = 0 \qquad (5)$$

and the Hamiltonian of a pure quartic oscillator is,

$$H = \frac{1}{2m}p_x^2 + \frac{1}{4}Kx^4 \qquad (6)$$

where x is the position and p_x is the corresponding momentum. K is the force constant.

Quartic oscillator model includes regular as well as chaotic systems. To study the chaotic behaviour we used a coupled quartic oscillator. The Hamiltonian for N=2 quartic oscillator is (Bannur, 1998; Bannur et al., 1997)

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}q_1^4 + \frac{1}{2}q_2^4 + \frac{\alpha}{2}q_1^2 q_2^2 \qquad (7)$$

where α is a parameter and q's and p's are generalized coordinates and momenta, respectively.

Van der Pol oscillator

Balthazar Van der Pol (van der Pol, 1926), a Dutch electrical engineer in his paper on "Relaxation oscillations" introduced a second order differential equation known as Van der Pol equation. The equation was similar to the one used with triode oscillators (Appleton & van der Pol, 1922; van der Pol, 1920). Van der pol started with a damped harmonic oscillator, then modified the system with the resistance as negative. In order to avoid the amplitude becoming infinity, Van der pol modified the coefficient of resistance. The coefficient is taken as a function of amplitude which will become positive for large values (van der Pol, 1926).

Van der pol equation became very popular quickly. Van der pol distinguished these oscillations as they differ from sinusoidal oscillations and he proposed the name relaxation oscillations. Now the Van der pol is used with a forcing term, which help to study the chaotic behavior (Ginoux & Letellier, 2012). The Van der Pol oscillator is governed by the following equation (van der Pol, 1926):

$$\boldsymbol{\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0} \qquad (8)$$

It describes many physical systems collectively called van der pol oscillators. The parameterμ is a positive scalar indicating the non linearity and strength of the damping. When μ = 0 there is no damping and the equation becomes,$\frac{d^2x}{dt^2} + x = 0$.

In the next section, we will describe each oscillator in detail, how and when they become chaotic.

Chaos a Pictorial Tour

We can see, in order to show the transition from periodic to chaotic behaviour we draw the phase space, time series and Poincare map of the above nonlinear oscillators. We used Python programs to draw the diagrams. In the case of Duffing oscillator and Van der pol oscillator we wrote the program using the differential equation of the oscillators and in the case of HenonHeiles oscillator and coupled quartic oscillator we wrote the program by using Hamilton's canonical equations derived from the Hamiltonian of the oscillator since there are no other differential equations.

Duffing Oscillator

Phase Space

We can see that on the first stage of oscillation, the system is in transient condition and there is no repetition in its motion. But after the initial transient, the motion is periodic and in Figure 1, we can see the periodic motion of a Duffing oscillator for γ= 0.30.

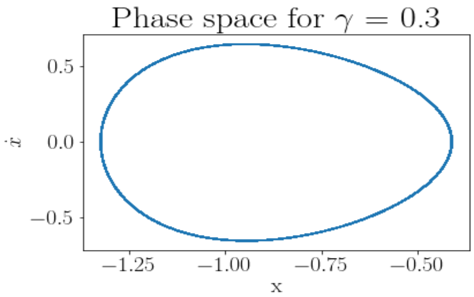

Figure 1. Phase space plot of Duffing oscillator for γ= 0.3

When $\gamma$is changed, then the period gets doubled for γ= 0.31,0.32,0.33 as shown in Figure 2.

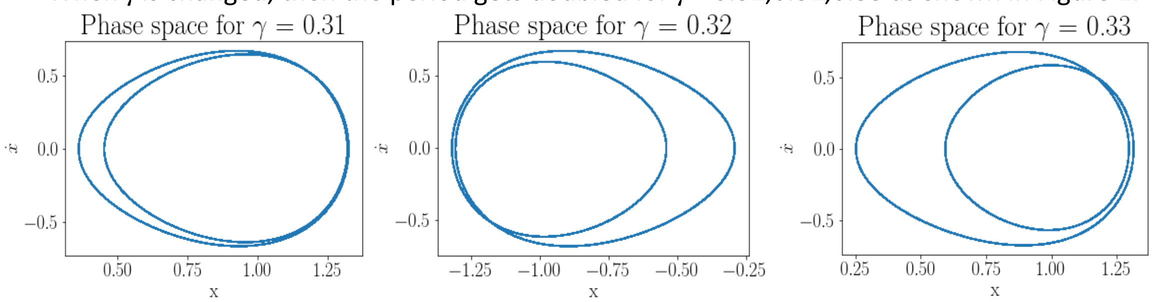

Figure 2. Phase space plots of Duffing oscillator for γ= 0.31, 0.32, 0.33

But when γ becomes 0.331 the periodicity is lost and there is no sign of the data settling down. This motion is chaotic which is shown in Figure 3.

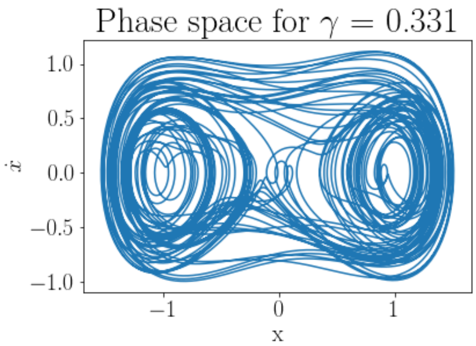

Figure 3. Phase space plot of Duffing oscillator for γ= 0.331

In Figure 4, the chaotic nature of Duffing oscillator is more evident for higher values of γ= 0.34, 0.35.

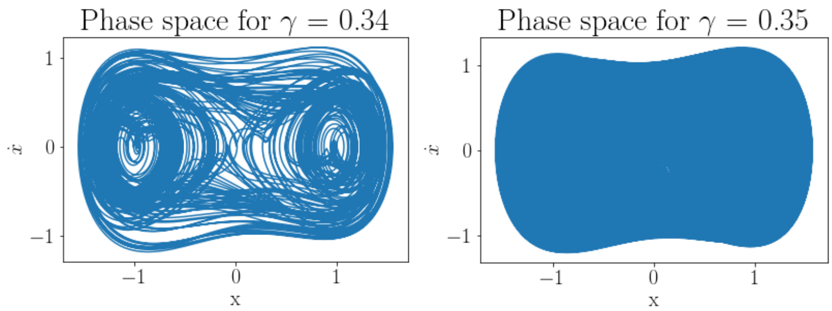

Figure 4. Phase space plots of Duffing oscillator for γ= 0.34, 0.35

Time Series

Time series showing the transition to chaos for various γvalues are plotted. In Figure 5 we can see the motion is periodic for γ = 0.30,0.31,0.32,0.33. and oscillator become chaotic when γ= 0.331 and more chaotic for γ= 0.34, 0.35 which is plotted in Figure 6.

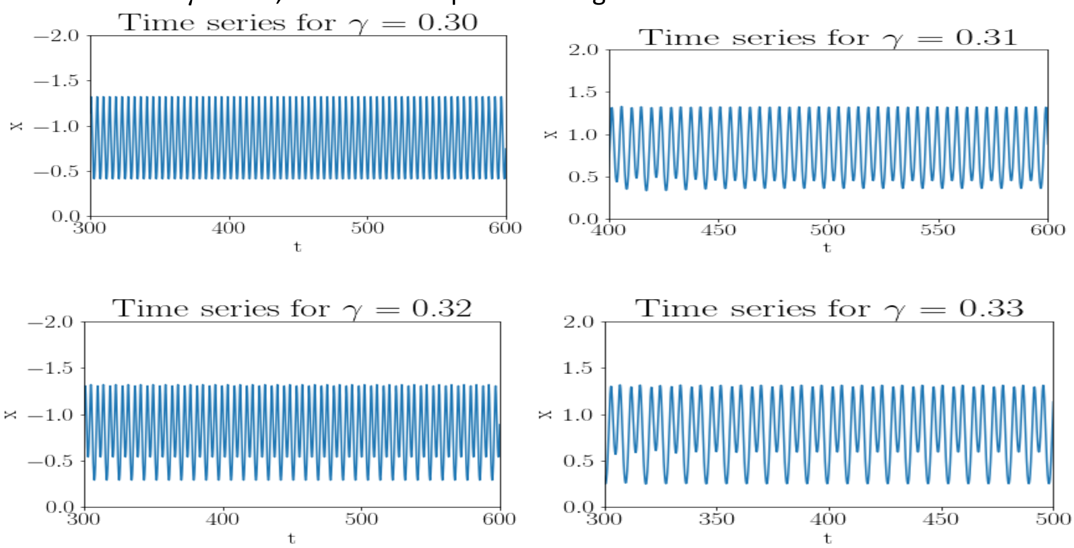

Figure 5. Time series plots of Duffing oscillator for γ= 0.30, 0.31, 0.32, 0.33

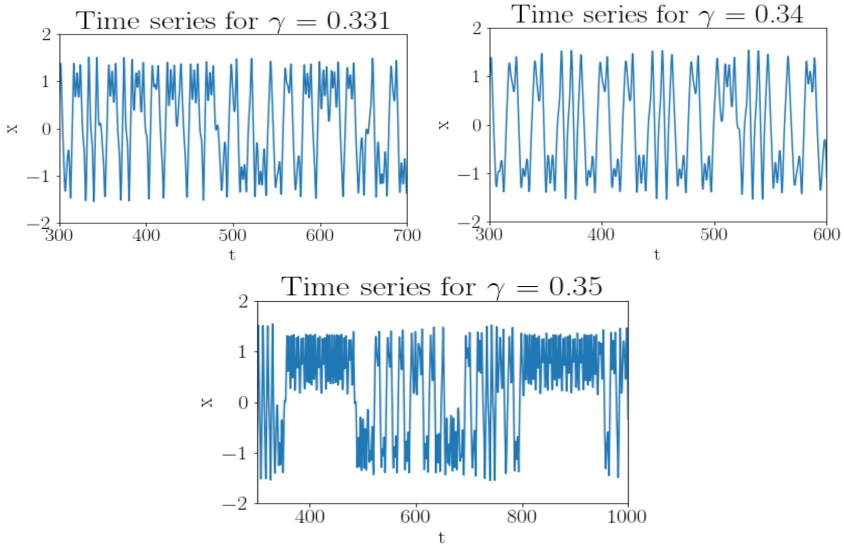

Figure 6. Time series plots of Duffing oscillator for γ= 0.331, 0.34, 0.35

Poincare Map

We can also confirm the chaotic character by drawing the Poincare maps of Duffing oscillator for various γvalues. There are only a few points in the map for γ= 0.31 and 0.33 as in Figure 7.

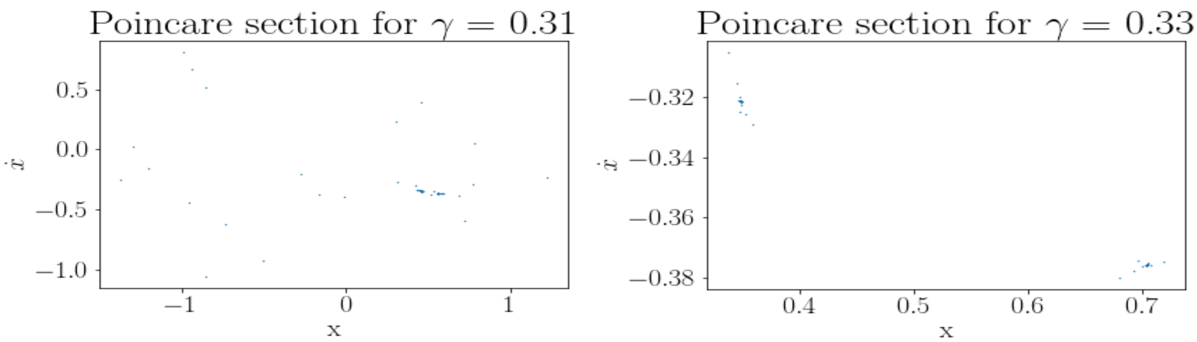

Figure 7. Poincare maps of Duffing oscillator for γ= 0.30, 0.31, 0.32, 0.33

We can see the transition to chaotic state in Figure 8 for γ= 0.331.

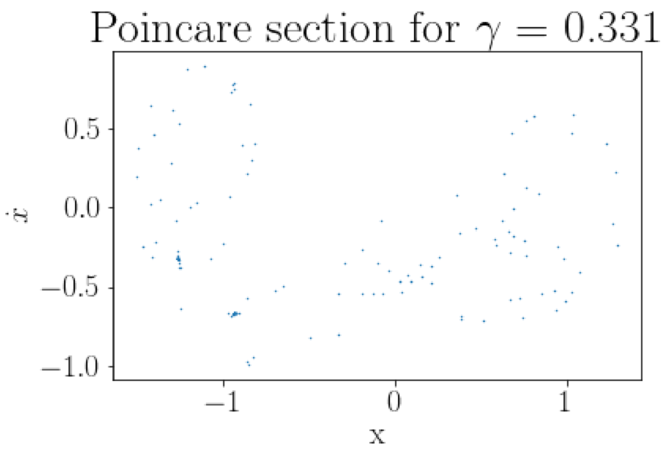

Figure 8. Poincare map of Duffing oscillator for γ= 0.331

For γ= 0.34, 0.35 a large number of points are there in the Poincare map which ascertains the chaotic behaviour which is shown in Figure 9.

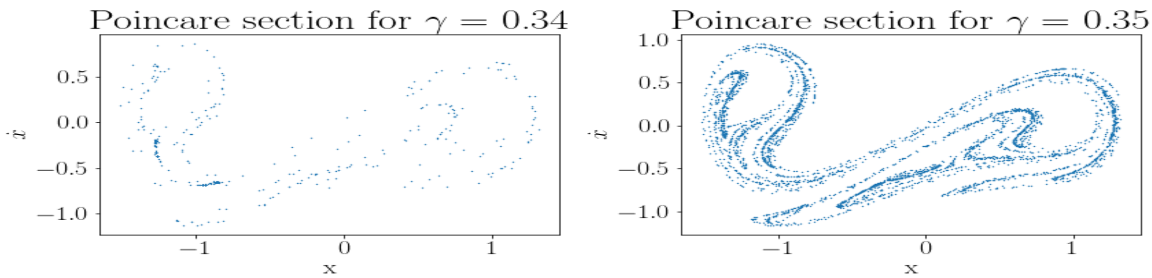

Figure 9. Poincare maps of Duffing oscillator for γ= 0.34, 0.35

Henon Heiles Oscillator

Phase Space

Here we plotted the phase space of Henon Heiles oscillator in Figure 10 for different energies and we can see the transition from normal to a chaotic state. For E =$\frac{1}{6}$it is chaotic.

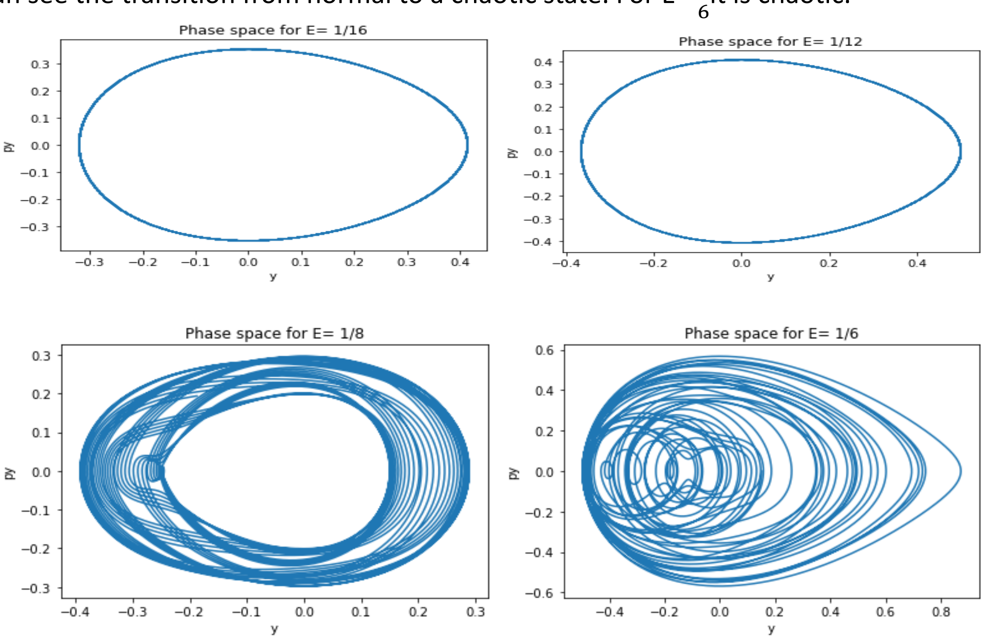

Figure 10. Phase space plots of HenonHeiles oscillator for E =$\frac{1}{16}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}$

One more graph is plotted, Figure 11, for energy E =$\frac{1}{5}(ie, E = 0.20)$and from these graphs we can say that Henon Heiles oscillator is chaotic for energy greater than$\frac{1}{6}(ie, E > 0.167)$.

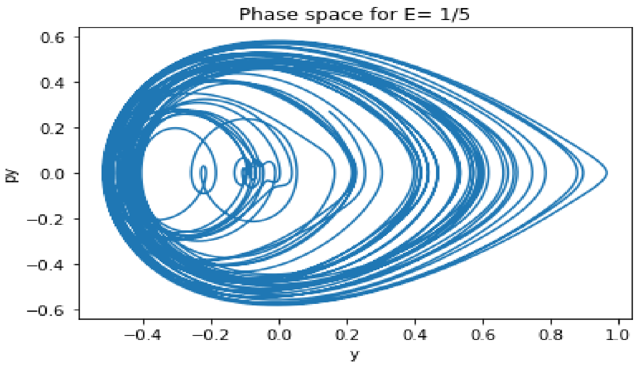

Figure 11. Phase space plot of HenonHeiles oscillator for E = $\frac{1}{5}$

Time Series

Time series showing the transition to chaos for different energies are plotted in Figure 12.

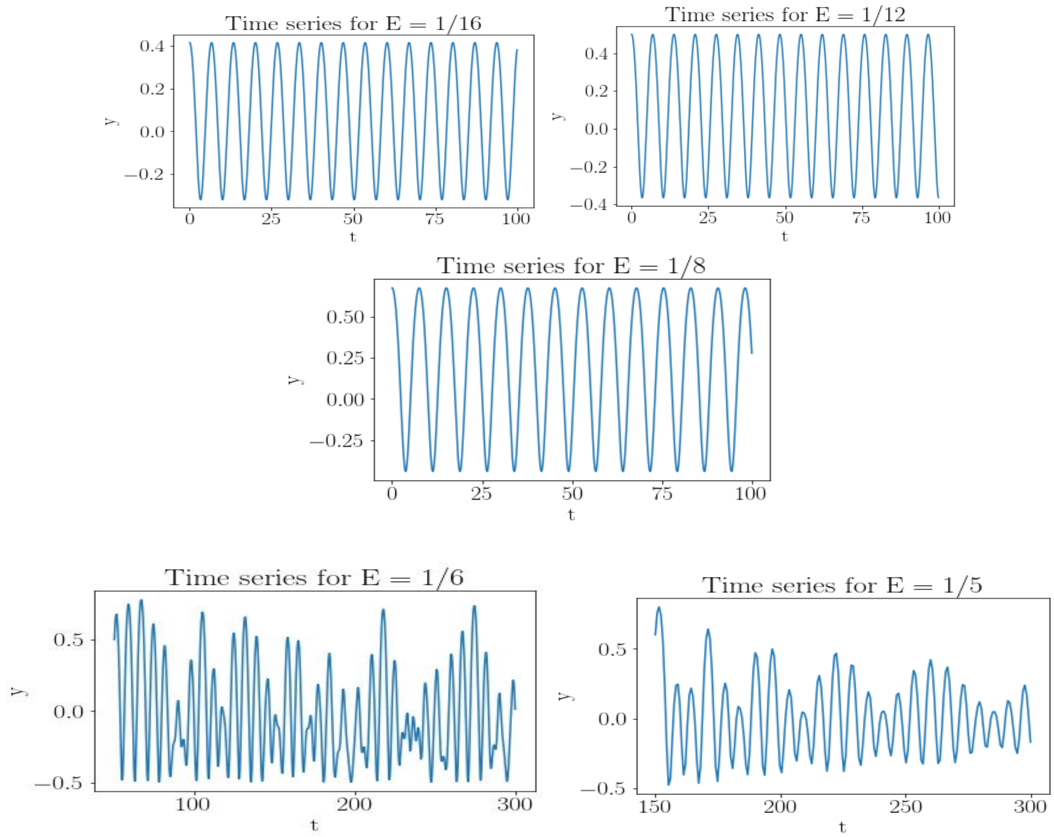

Figure 12. Time series plots of HenonHeiles oscillator forE =$\frac{1}{16}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{1}{5}$

Poincare Map

Poincare maps are also plotted for the same set of energies in Figure 13.

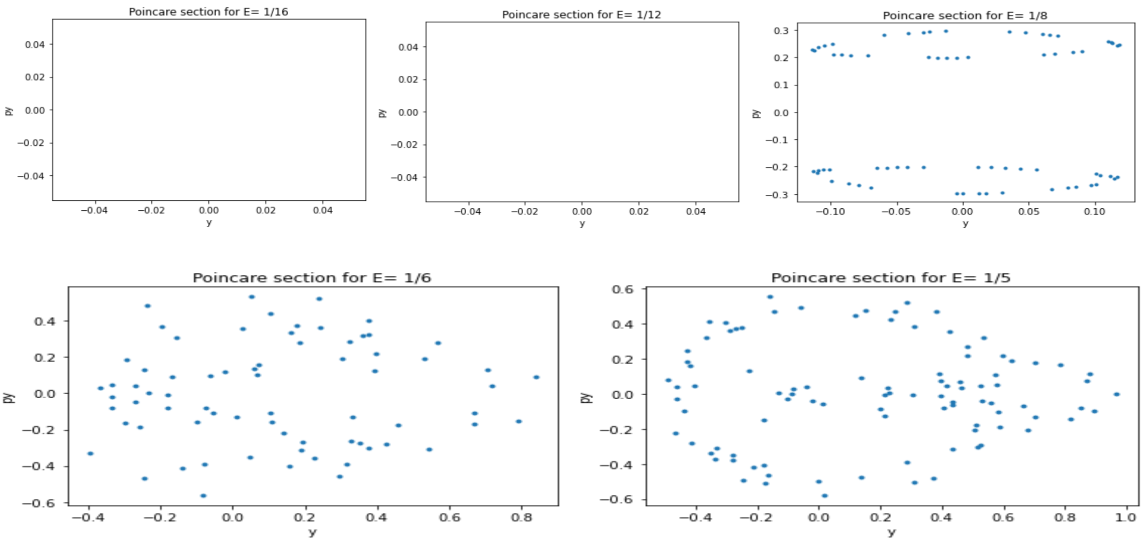

Figure 13. Poincare maps of HenonHeiles oscillator forE =$\frac{1}{16},\frac{1}{12},\frac{1}{8},\frac{1}{6},\frac{1}{5}$

Quartic oscillator

Phase Space

The phase space diagrams of quartic oscillator are plottedfor α = 0,2 in Figure 14 and the motion is periodic.

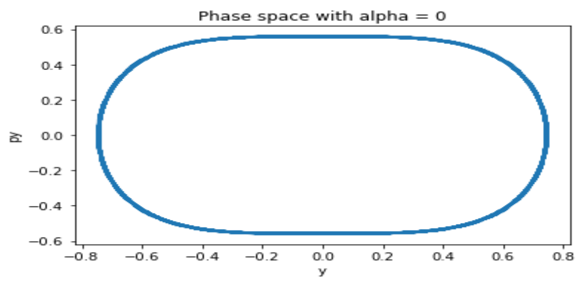

Figure 14. Phase space plot of Quartic oscillator forα = 0

The chaotic behavior can be observed for $\alpha > 6$and is very clear for high values of α(α= 100, 500) as shown in Figure 15.

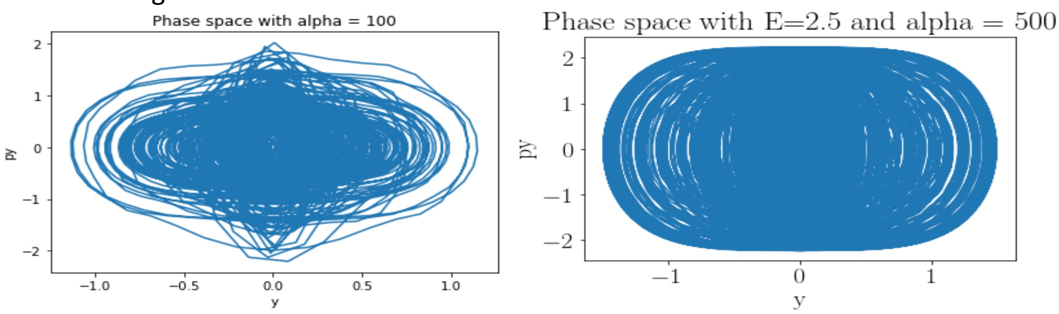

Figure 15. Phase space plot of Quartic oscillator forα = 100, 500

Time Series

The time series of quartic oscillator for α = 0 is plotted in Figure 16 and for α = 100, 500 plotted in Figure 17.

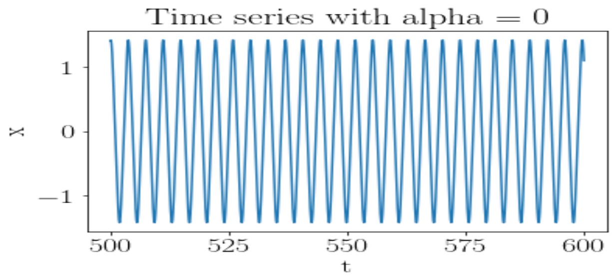

Figure 16. Time series plot of Quartic oscillator forα = 0

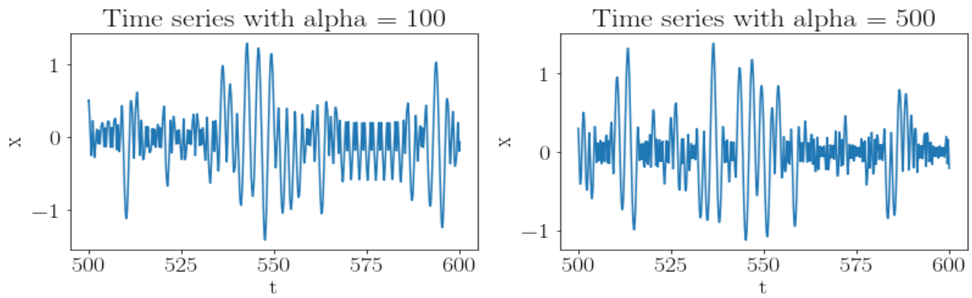

Figure 17. Time series plot of Quartic oscillator forα = 100, 500

Poincare Map

The Poincare map for α = 0 is shown in Figure 18 and for α = 100, 500 are shown in Figure 19 which clearly explain the transition from periodic to chaotic behaviour.

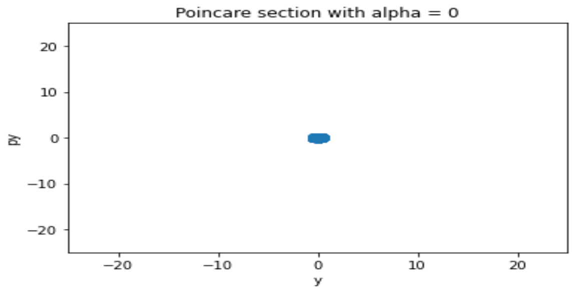

Figure 18. Poincare map of Quartic oscillator forα = 0

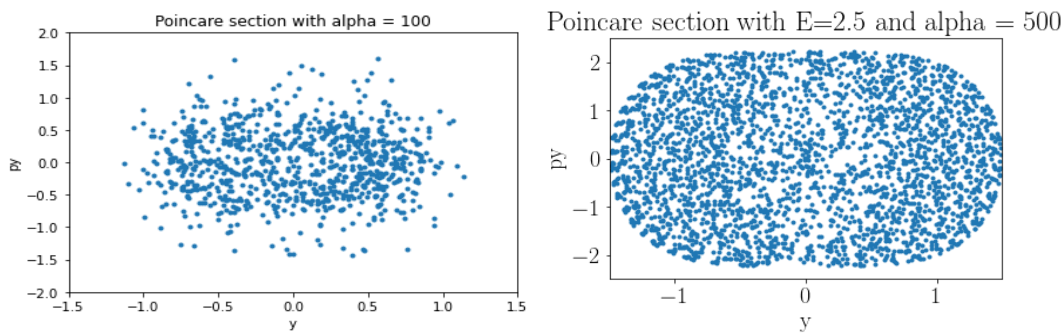

Figure 19. Poincare maps of Quartic oscillator forα = 100, 500

Van der pol oscillator

Here we shown the chaotic nature of Van der pol oscillator only to some extent because of the low processing speed of the computer. With high speed computers we can draw the plots for larger time interval which can show the chaotic behavior more explicitly.

Phase Space

The phase space diagram for μ = 0.01,10,1000 are plotted in Figure 20 which is chaotic.

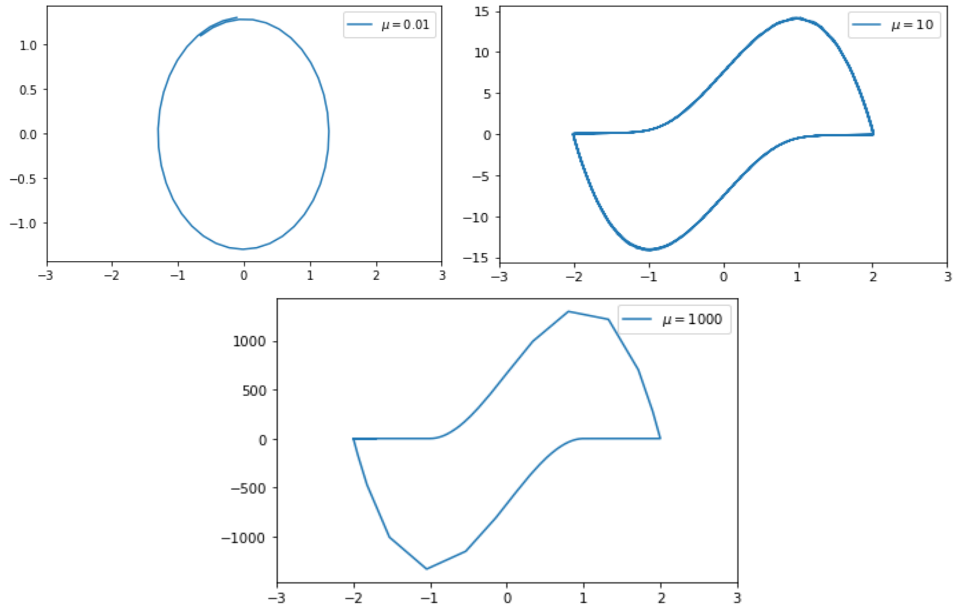

Figure 20. Phase space plots of Van der pol oscillator forμ = 0.01,10,1000

Time Series

Here we confirm the chaotic nature of Vander pol oscillator by drawing the time series for μ = 0.01,10,1000 in Figure 21. Now let us have a brief idea about fractals in the next section.

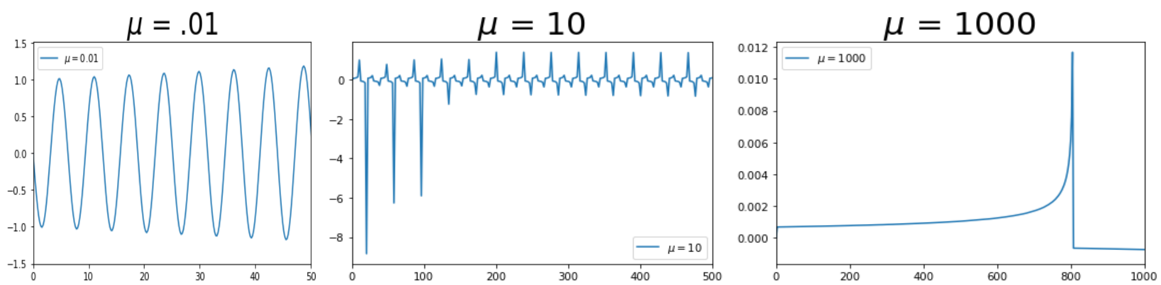

Figure 21. Time series plots of Van der pol oscillator forμ = 0.01,10,1000

Fractals

A fractal is an object that looks similar in different scales. In nature many objects that has an irregular shape in normal view has repeated patterns which can be seen on magnification. The word fractal was given by Benoit Mandelbrot in 1975 (Mandelbrot, 1982). It was obtained from the Latin word fractus meaning fractured. Mathematically a fractal is obtained by the process called iteration. Practically different parts of the object are removed repeatedly to form a fractal structure.

Fractal dimension

We live in a 3 -dimensional world. Look at the Figure 22. The mesh in background has squares with unit length for their sides. We have three objects a line, a square and a cube which are named a, b and c. Here our straight line has length of 3 units. If we divide the straight line to parts, each with length equal to side length of the mesh square, we get three pieces of straight lines. To make the original line of full length we have to enlarge each small line to three times. Hence we can write a relation for the dimension of objects as

Number identical pieces = (incremental factor or scaling factor)^D

where D is the dimension of the object.

For the line a we get $3=3^D$.This indicates that D=1.

For the square b we get in similar manner $9=3^D$. This indicates that for square we have D=2.

For the cube c we get $27=3^D$. In this case D=3

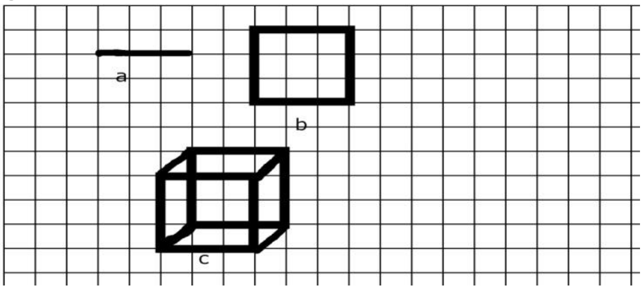

Figure 22. Different shapes

Thus generally we can say if n is the number of identical pieces and p is the scaling factor, we can write

$$n = p^D$$

Taking logarithm, we get expression for dimension as

$$D = \frac{\log n}{\log p}$$

The dimension in this form are called similarity dimension as the structures are considered as identical pieces to calculate the dimension. To characterize the trajectories of dynamical systems a spectrum of dimensions are introduced and box counting dimension, correlation dimension and information dimension are some among them that gained wide popularity(Mou et al., 2016; Rosenberg, 2020). We will illustrate the box counting dimension in the next section using which we are going to find the fractal dimension for some chaotic systems.

Box-counting dimension

This is the most popular and commonly used method to calculate the fractional dimension (Mou et al., 2016; Thompson, 2016). In this method identical square boxes of side length k is placed on the object whose fractal dimension has to be found. The number of boxes n needed to cover the entire object is found and fractal dimension is found from the relation

$$D_B = \lim_{k \to 0} \frac{\log n}{\log \frac{1}{k}}$$

In practice, a plot of the number of identical boxes on a log-log graph and the scale gives a straight line, whose slope gives the fractal dimension.

Fractal dimension of non-linear systems

To find the fractal dimension of non-linear systems 2D plots of phase space diagrams that are plotted are used. The figures are first saved in a bitmap format. Then the dimension is found using Fractalyse software. Fractalyse is a free software available on the internet designed to find the fractal dimension of 2D patterns. The software was developed by G. Vuidel and co-workers in France. The software is widely used to study the fractal structure of different shapes. The software can be easily installed in a computer by downloading the executable file. The images are used in bitmap format for the analysis. To find the dimension, the images are opened from file menu and box counting method is chosen from analyse menu. The specific options used are

File→ open

Analyse → Box

The new window emerged by this option gives the plot the number of boxes required to cover the object and scaling factor in a logarithmic scale.

Non-linear oscillators like Duffing, Henon- Heiles, Quartic and Van der pol exhibit chaotic behavior in their phase portrait at certain values (Tarnopolski, 2013). We found the fractal dimension of the above four oscillators using Fractalyse software when they are chaotic. For Duffing oscillator when γ = 0.331, fractal dimension is 1.798. For a Henon Heiles oscillator with energy, E = $\frac{1}{6}$ fractal dimension is 1.738. For the phase space plot of Quartic oscillator at α = 100 we got the fractal dimension of 1.78. In the case of a Van der pol oscillator fractal dimension is 1.331 for μ= 1000.

The fractal dimension of regular geometrical shapes matches with the Euclidean dimension, the fractal dimension of a line is 1 and the fractal dimension of a square is 2. A curve with fractal dimension 1.08 behaves like a line because the dimension is close to 1. The fractal dimension of a circle, which can be treated as an extended curve, is 1.15 which is not a good fractal.

The fractal dimensions of the nonlinear oscillators we have discussed are shown in Table 1, which shows their fractal character. Here all the oscillators are having fractal dimension greater than 1.5 except Van der pol oscillator. But for Van der pol oscillator if we can plot the phase space for large values of time, we anticipate that the fractal dimension will be more than 1.5. Hence we can say that all the four oscillators are good fractals.

Table 1. Dimensions of the Nonlinear Oscillators

Serial No.	Oscillator	Fractal dimension D
1	Duffing	1.798
2	HenonHeiles	1.738
3	Quartic	1.78
4	Van der pol	1.331

Conclusion

In this article we have made a brief description about chaos and the methods for detecting chaos. Chaos are observed usually in nonlinear differential equations. So we explained the difference between linear and nonlinear differential equations. Then we took four nonlinear oscillators and by using Python program plotted the phase space, time series and Poincare sections of all these oscillators which clearly explained the chaotic behaviour of the oscillators at some critical values. Then we gave a brief discussion about fractals which exhibits interesting features like self-similarity, but have only pictorial origin and usually mathematical explanations are difficult. By using Fractalyse software we computed the fractal dimension of the four nonlinear oscillators when they are chaotic. All of them have good fractal dimension values. So they are fractals. Thus we established that chaotic systems are fractals. Through this article we had given a simple method to show that all chaotic systems are fractals, even though the pictorial representation of chaotic systems does not match with the popular fractals available in nature.

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